Calculus Group I (1)

Two straight hallways meet at right angles. See figure below. The first hallway has width $A$, and the second hallway has width $B$. A ladder of negligible thickness is to be taken horizontally from one hallway to the other. Find the length of the longest such ladder.

Calculus Group I (2)

Suppose that the points $P, Q, R$ have coordinates $(0, 5), (0, 3), (x, 0)$ respectively. Find the value or values of $x$ which maximize the size of angle $PRQ$.

Calculus Group I (3)

Let $f(x) = \int_1^x \frac{1}{1 + t^6} dt$. Evaluate the integral $\int_0^1 xf(x) dx$.

Calculus Group I (4)

A baseball, hit by a baseball player at a $25^\circ$ angle from 3 ft above the ground, just cleared the left-field wall. This wall is 35 ft high and 320 ft from home plate. Use the Ideal Projectile Motion Equation (in vector form)

$$r(t) = \left((v_0 \cos \alpha) t + x_0\right)i + \left(-\frac{1}{2}gt^2 + (v_0 \sin \alpha) t + y_0\right)j$$

to answer the following questions. Note that $g = 32 \text{ ft/sec}^2$, and that $i$ and $j$ are the standard unit vectors: $i = (1, 0), j = (0, 1)$.

(a) How long did it take the ball to reach the wall?
(b) What was the initial speed of the ball?
(c) Would the same hit have cleared the right field wall which is 5 ft high and 420 ft from home plate? (show work)
(d) For this hit, what is the position vector of the ball at its maximum height?

Calculus Group I (5)

A curve in the $xy$-plane is defined by the parametric equations

$$x(t) = t^2, \quad y(t) = t^3 - 3t.$$ 

This curve intersects itself at the point $(3, 0)$, and so it makes a self-enclosed loop as $t$ ranges from $-\sqrt{3}$ to $\sqrt{3}$.

(a) Find all points on this loop that have either horizontal or vertical tangent lines.
(b) Calculate the area enclosed by the loop.
**Abstract Algebra Group I (6)**

The general linear group $GL_2(\mathbb{R})$ is the group of $2 \times 2$ invertible matrices with real entries. The group operation is ordinary multiplication of matrices. Let $I$ denote the $2 \times 2$ identity matrix and let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

(a) Find the orders of $A$ and $B$ as elements in the group $GL_2(\mathbb{R})$.

(b) Verify that $BA = AB^2$.

(c) Explain why $H = \{I, B, B^2, A, AB, AB^2\}$ is a subgroup of $GL_2(\mathbb{R})$. Construct a Cayley table for $H$ as part of your explanation.

(d) Find the number of nonisomorphic subgroups of $H$.

**Advanced Calculus Group I (7)**

Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, n \geq 1$.

Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

**Differential Equations Group I (8)**

Consider the population model

$$\frac{dP}{dt} = 2P \left(1 - \frac{P}{100}\right) \left(\frac{P}{10} - 1\right),$$

where $P \geq 0$ is the population size and $t$ is time in years.

(a) Suppose that the initial population is two hundred, that is $P(0) = 200$. What is the long-term behavior of the population in this case?

(b) Suppose that the initial population is fifty, that is $P(0) = 50$. What is the long-term behavior of the population in this case?

(c) Describe all possible behaviors of the population.

**Discrete Mathematics Group I (9)**

Strings (sequences of letters) are formed using the alphabet $\{A, B, C, D, E\}$.

(a) How many strings are possible of length eight?

(b) How many strings of length eight include the letter E exactly once?

(c) How many strings of length eight include the letter E at least once?

(d) How many strings of length eight have no consecutive letters the same?

(e) How many strings of length eight are composed of exactly two letters from the alphabet?
Linear Algebra Group I (10)

An $n \times n$ matrix with real entries $A$ is called idempotent if $A^2 = A$. Let $p_A(t) = \det(tI - A)$ denote the characteristic polynomial of $A$.

For each of the following, either prove or provide a counterexample.

(a) $A$ is idempotent implies that $A$ is invertible.
(b) An idempotent matrix $A$ can have at most two distinct eigenvalues.
(c) If $p_A(t) = t^2 - t$, then $A$ must be idempotent.
(d) If $p_A(t) = t^3 - 2t^2 + t$, then $A$ must be idempotent.

Abstract Algebra Group II (11)

The quaternion group, denoted $Q$, can be represented as

$Q = \{I, \pm i, \pm j, \pm k\}$,

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},$$

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},$$

and where the group operation is ordinary matrix multiplication.

Note: In this problem $i \neq \sqrt{-1}$.

(a) Using the above matrix notation, decide whether or not $Q$ is abelian. Justify your answer.
(b) Construct the Cayley table for $Q$.
(c) Is $Q$ a simple group? Justify your answer.

Linear Algebra Group II (12)

Let $n > 1$ be a positive integer, and let $A$ be an $n \times n$ matrix with real entries. Show that if $\text{rank}(A) = k$, then $A$ has at most $k + 1$ distinct eigenvalues. Give an example where $A$ has exactly $k + 1$ eigenvalues, to show this is best possible.

Advanced Calculus Group II (13)

Let $f : [0, \infty) \to [0, \infty)$ and $f(0) = 0$. Consider the integral equation $f^{-1} = \int f$, along with the differential equation $f' = 1/(f \circ f)$.

(a) Find necessary conditions on $f$ so that the equations are both well posed.
(b) Prove that the two equations are equivalent.
(c) If $f(x) = kx^a$ is a solution to the two equations, where $k > 0$ and $0 < a < 1$, what must $a$ equal?
Advanced Calculus Group II (14)

Suppose that \( \{a_k\}_{k=1}^{\infty} \) is a sequence of positive real numbers and that \( \sum_{k=1}^{\infty} a_k < 3. \)
Prove that \( \sum_{k=1}^{\infty} \frac{\sqrt{a_k}}{2^k} < 1. \)

Differential Equations Group II (15)

The temperature \( T \), measured as a percentage of 100° F, in a certain room on a cold day varies with time \( t \), measured in hours. When the heating unit is ON,
\[
\frac{dT}{dt} = 1 - T
\]
and when the heating unit is OFF,
\[
\frac{dT}{dt} = -T.
\]
If the heating unit is ON, and the temperature is greater than or equal to 75° (\( T \geq 0.75 \)), the heating unit switches OFF. If the heating unit is OFF and the temperature is less than or equal to 65°, the heating unit switches ON.

At 9:00 AM, \( T = 0 \) and the heat is switched ON.
(a) At what time does the heat switch off for the first time? Round your answer to the nearest minute.
(b) Find the temperature at 10:30 AM. Round your answer to the nearest degree.

Multivariable Calculus Group II (16)

Dr. E. Ville keeps a pet shark, with a laser on its head, in a pool of water which is 50 m wide by 50 m long by 20 m deep. The laser beam points straight forward in the direction in which the shark is travelling.

Dr. Ville imposes a coordinate system, with the origin at the point 10 m deep in the center of the pool, the \( x \)-axis perpendicular to the eastern wall of the pool, the \( y \)-axis perpendicular to the northern wall, and the positive \( z \)-axis intersecting the the surface of the pool.

He observes his shark for 5 seconds (\( 0 \leq t \leq 5 \)) and finds that the position of his shark is given by the vector
\[
r(t) = t^2 i + (\sin t - t \cos t) j + (\cos t + t \sin t) k.
\]
(a) What is the shark’s velocity when \( t = \pi \)?
(b) How far does the shark travel during the interval \( 0 \leq t \leq 5 \)?
(c) At \( t = \pi / 3 \), what does the laser beam hit: a wall, the surface of the water, or the floor of the pool? If it hits a wall, which wall does it hit?
Geometry Group II (17)

In a certain right triangle, the distance between the points where the altitude and the median meet the hypotenuse is one-third the length of the hypotenuse. Find the ratio, larger to smaller, of the lengths of the legs of this right triangle.

Topology Group II (18)

Let \((X, \mathcal{I})\) be a topological space. We say \((X, \mathcal{I})\) is a \(T_1\)-space provided that, for distinct \(a, b \in X\), there are \(U, V \in \mathcal{I}\) so that \(a \in U, b \notin U, a \notin V, \) and \(b \in V\). We say \((X, \mathcal{I})\) is a \(T_2\)-space provided that, for distinct \(a, b \in X\), there are \(U, V \in \mathcal{I}\) so that \(a \in U, b \in V\) and \(U \cap V = \emptyset\).

(a) If \((X, \mathcal{I})\) is \(T_2\), prove that all convergent sequences have unique limits.

(b) If all convergent sequences of \((X, \mathcal{I})\) have unique limits, prove that \((X, \mathcal{I})\) must be \(T_1\).

(c) Let \(X = \mathbb{R}\) and let

\[\mathcal{I} = \{U \mid U \subseteq X \text{ and } X \setminus U \text{ is countable}\} \cup \{\emptyset\}.\]

Prove that convergent sequences of \((X, \mathcal{I})\) have unique limits and that \((X, \mathcal{I})\) is not \(T_2\).

[Note that \(X \setminus U\) is the complement of \(U\) in \(X\).]