Let $a_n > 0$ for all $n \in \mathbb{N}$, and let $\sum_{n=1}^{\infty} (-1)^{n+1}a_n$ be a conditionally convergent alternating series. Let $a^+ = \{a_1, a_3, a_5, \ldots\}$ be the set of positive terms of this alternating series, and let $a^- = \{-a_2, -a_4, -a_6, \ldots\}$ be the set of negative terms of the alternating series. Since $\sum_{n=1}^{\infty} (-1)^{n+1}a_n$ is conditionally convergent, $\sum_{n=1}^{\infty} a_{2n+1} = +\infty$ and $\sum_{n=1}^{\infty} -a_{2n} = -\infty$. An ordered rearrangement of $\sum_{n=1}^{\infty} (-1)^{n+1}a_n$ is an infinite series which begins by adding a finite number of terms from $a^+$ in order, followed by adding a finite number of terms from $a^-$ in order, and so on. For example, one ordered rearrangement of $\sum_{n=1}^{\infty} (-1)^{n+1}a_n$ is:

$$(a_1 + a_3) + (-a_2 - a_4 - a_6) + (a_5 + a_7 + a_9) + (-a_8 - a_{10}) + \ldots$$

Let $S > 0$. Describe how to construct an ordered rearrangement of $\sum_{n=1}^{\infty} (-1)^{n+1}a_n$ whose sum is $S$.

**Calculus Group I (2)**

Let

$$f(x) = \int_0^x [t] \, dt \quad \text{and} \quad g(x) = \int_x^1 \lfloor 1/t \rfloor \, dt,$$

where $[t]$ denotes the greatest integer $\leq t$.

(a) Sketch the graph of $f$ on the interval $[0, 3]$.

(b) Find the exact value of $f(2^{2008} + 1)$.

(c) Let $n$ denote an integer. Find $\lim_{n \to \infty} g(1/n)$. Is the improper integral $\int_0^1 \lfloor 1/t \rfloor \, dt$ convergent or divergent? Justify your answer.

**Calculus Group I (3)**

Let $S$ be the region in the plane bounded by the following curves:

$$x = 0, \quad y = \sqrt{6x - x^2}, \quad y = \frac{\sqrt{3}}{3}x - 2\sqrt{3}.$$ 

Find the volume of the solid generated by rotating $S$ about the $x$-axis.

**Calculus Group I (4)**

Give a MacLaurin series expansion for the function $h(x) = x^3 \tan^{-1} x$.

**Calculus Group I (5)**

A spherical balloon is being inflated so that its volume increases at a constant rate of 5 cubic centimeters per second. How fast is the surface area of the balloon increasing at the instant when the surface area is 40 square centimeters?
Abstract Algebra Group I (6)

(a) Let $G = \langle a \rangle = \{e, a, a^2, a^3, a^4, a^5\}$ be the cyclic group of order 6. List all subgroups of $G$.

(b) Prove or disprove: If a finite group, with more than one element, has no nontrivial subgroups, then the order of the group must be prime.

(c) Prove or disprove: Any group with more than one element and no nontrivial subgroups must be finite and the order of the group must be prime.

Advanced Calculus Group I (7)

Suppose that $f$ is a continuous, real-valued function on the closed interval $[a, b]$. Prove that $f$ is uniformly continuous.

Differential Equations Group I (8)

Suppose a tank initially holds 500 L of salt water with a concentration of 2 kg/L. There are two input valves on the tank. For the first 10 minutes, valve A is open and empties 12 L/min of brine containing 4 kg/L salt. After 10 minutes, valve A is closed and valve B begins emptying 12 L/min of brine containing 6 kg/L salt. The exit valve C removes 12 L/min at all times, keeping the volume constant. Find the amount of salt in the tank at any time $t \geq 0$.

Discrete Mathematics Group I (9)

A student club must send at least one representative to an open house for prospective students. The members of the club decide to randomly select who will attend the open house. The club has a total of 13 members, one of whom is Samantha. Give exact answers rather than decimal values.

(a) What is the probability that Samantha is included in the group at that open house?

(b) What is the probability that the club will send just one member to the open house?

Linear Algebra Group I (10)

Let $P_2(\mathbb{R})$ denote the vector space of polynomials of degree $\leq 2$ with real coefficients and define $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$(Tp)(x) = 2p(x) - \frac{1}{x^2 - 1} \int_1^x p'(t) \, dt.$$ 

(a) Find the matrix of $T$ relative to the basis $\{1, x, x^2\}$.

(b) Determine the eigenvalues of $T$.

(c) Is $T$ diagonalizable? (Justify.)
Abstract Algebra Group II (11)

Let \( \mathbb{Z} \) be the set of integers and let \( \mathbb{Q} \) be the set of rational numbers. View both as abelian groups under the usual addition operation. Recall that if \( H \) is a subgroup of an abelian group \( G \), then the index of \( H \) in \( G \) is the number of distinct cosets of \( H \) in \( G \).

(a) (3 points) Prove that for any integer \( n \geq 2 \), \((\mathbb{Z},+)\) has a subgroup of index \( n \).
(b) (7 points) Prove that \((\mathbb{Q},+)\) has no proper subgroup of finite index.

Linear Algebra Group II (12)

Let \( V \) be a vector space and let \( W \subset V \). \( W \) is a subspace of \( V \) if and only if \( W \) is closed under addition and scalar multiplication. Let \( M_{n,n} \) be the set of all \( n \times n \) matrices. \( M_{n,n} \) is a vector space under the usual operations of matrix addition and scalar multiplication.

(a) An \( n \times n \) matrix \( A \) is called symmetric if \( A^T = A \). Let \( S_{n,n} \) be the set of \( n \times n \) symmetric matrices. Note that \( S_{n,n} \subset M_{n,n} \). Show that \( S_{n,n} \) is a subspace of \( M_{n,n} \).
(b) An \( n \times n \) matrix \( A \) is called singular if \( A \) is not invertible. Let \( T_{n,n} \) be the set of \( n \times n \) singular matrices. Note that \( T_{n,n} \subset M_{n,n} \). Show that \( T_{n,n} \) is not a subspace of \( M_{n,n} \).

Advanced Calculus Group II (13)

For each integer \( n \geq 1 \), define the polynomial \( f_n(x) = x^n + x - 1 \).

(a) (2 points) Prove that, for all integers \( n \geq 1 \), \( f_n(x) \) has a unique positive real root \( p_n \).
(b) (8 points) Find \( \lim_{n \to \infty} p_n \), and prove your claim.

Advanced Calculus Group II (14)

Recall the metric space \((X,d)\) is bounded if there exists a number \( L \) so that \( d(x,y) < L \) for all \( x \) and \( y \) in \( X \), and \((X,d)\) is complete if every Cauchy sequence in \( X \) has a limit in \( X \). (The sequence \( x_1, x_2, x_3, \ldots \) is Cauchy if for each positive number \( \varepsilon > 0 \), there exists an integer \( N \) so that if \( n > N \) and \( m > N \), then \( d(x_n, x_m) < \varepsilon \)).

Recall a space \( X \) is compact if every covering of \( X \) by open sets has a finite subcover.

(a) Prove directly or find a counterexample: Every compact metric space is complete and bounded.
(b) Prove directly or find a counterexample: Every complete and bounded metric space is compact.
Differential Equations Group II (15)

Consider the population model
\[
\frac{dP}{dt} = \frac{P}{3} \left(1 - \frac{P}{100}\right) \left(\frac{P}{10} - 1\right),
\]
where \(P(0) = P_0\) = the initial population (which is assumed to be positive), \(P\) is the population, and \(t\) is the time in years.

a.) Describe all possible behaviors of the population.

b.) Suppose that the initial population is \(P_0 = 20\). Find (or estimate) the population in year 1. What is the long term behavior of the population in this case?

Multivariable Calculus Group II (16)

\(Q\) is a strange shape in the \((x, y)\)-plane determined by the following formulas:
If \(0 < u < 1\) and \(0 < v < 1\) then \((x, y)\) is in the set \(Q\) if \(x = e^u + v^2\) and \(y = e^v + v\).

Find the area of \(Q\).

Geometry Group II (17)

Let \(v_1, v_2, v_3, \ldots, v_9\) be the vertices of a regular nonagon, listed in the counterclockwise direction. Let \(P\) be the point of intersection of the lines \(\overrightarrow{v_1v_2}\) and \(\overrightarrow{v_5v_6}\). Find the measure of \(\angle v_2Pv_5\).

Topology Group II (18)

For any topological space \(X\) we have a diagonal map
\[
\Delta : X \to X \times X \text{ defined by } \Delta(x) = (x, x).
\]
Show that the space \(X\) is Hausdorff if and only if the image of the diagonal map is a closed subset of \(X \times X\).

Probability Group II (19)

Richard, Jake, Lisa, and Esme are all playing a hand of bridge. Bridge is a card game played with a standard deck of 52 cards. In a given hand, each player is dealt 13 cards, that is the entire deck is split evenly among all four players. If it is known that Richard has exactly one ace, and that Jake has exactly one ace, what is the chance Esme has both the remaining two aces?

Set Theory/Logic Group II (20)

Let \(A\) and \(B\) be sets and let \(p : A \times B \to B\) be the function defined by \(p(a, b) = b\).
Prove or disprove: \(p\) is a surjection.